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COMMENT

Mean field–finite size scaling transformations for the Z_4 spin model

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Abstract. The recursion relations which result from the scaling law for the magnetisations of two finite systems surrounded by the mean field are obtained for the Z_4 spin model. Then, following a renormalisation group strategy, the phase diagrams and critical exponents are determined and studied. This method leads to a substantial improvement, at least for determining phase diagrams, with respect to mean field approximation or other renormalisation group techniques.

In this paper is applied the method proposed by Indekeu, Maritan and Stella (1982), henceforth referred to as IMS, to a two coupling constants model: the Z_4 spin model, in two and three dimensions. This method has been applied to random systems (Droz *et al* 1982) and to the triangular Ising ferromagnet (Slotte 1983).

As in finite size scaling theory (Nightingale 1976), the IMS method is based on the comparison of systems of different sizes. For both systems, the average magnetisation can be computed in the presence of symmetry breaking boundary conditions which, in a mean field (MF) sense, simulate the effect of an infinite lattice. The imposition of the standard scaling law between the magnetisations which arises in the real space renormalisation group (RSRG) transformations (Niemeijer and van Leeuwen 1976) leads after linearisation to a recursion relation for the parameters of the model.

By making use of these recursions as the RG theory prescribes, the phase diagrams of the model and its critical indices can be obtained. The formal connection between the IMS approach and more standard RSRG methods has been indicated in the original paper and we shall not repeat it here. The application of the IMS method to the well studied two-dimensional Z_4 spin model is performed in order to check its capabilities. Then it is used for determining the phase diagram of the less known three-dimensional Z_4 spin model.

The most general action with a Z_4 global symmetry, in the presence of external fields, is

$$A[\sigma]=J_1\sum_{\langle i,j\rangle}\mathrm{Re}(\sigma_i\sigma_j^*)+J_2\sum_{\langle i,j\rangle}\sigma_i^2\sigma_j^2+h_1\sum_i\mathrm{Re}\,\sigma_i+h_2\sum_i\sigma_i^2 \qquad (1)$$

where the spin variables $\sigma_i=+1,-1,+\mathrm{i},-\mathrm{i}$, are placed in the sites of a d-dimensional hypercubic lattice. $\langle i,j\rangle$ indicates nearest neighbours.

Following the process sketched above, two finite systems will be considered: a single spin and a square of two sites per side. In $d=3$, this cluster is somewhat more isotropic than the one adopted by IMS for the analysis of the Potts model.

In order to recover all the phases of the model, the original IMS prescription has to be slightly generalised by introducing a two-component order parameter (f, g) where $f=\langle\sigma_i\rangle$ and $g=\langle\sigma_i^2\rangle$. The corresponding parameters of the $2d$ surrounding spins of the single spin are fixed to the mean field values v' and u'. Similarly, each of the four spins of the cluster has 2(4) boundary spins for $d=2(3)$, that are fixed to different MF values v and u. The next step is to compute the order parameters for the single spin and for the cluster. Then the fundamental equations of the method, obtained by connecting both order parameters through the usual finite size scaling relation, are

$$f'(J_1', J_2'; h_1', h_2'; v', u') = 2^{-a} f(J_1, J_2; h_1, h_2; v, u),$$
$$g'(J_1', J_2'; h_1', h_2'; v', u') = 2^{-2a} g(J_1, J_2; h_1, h_2; v, u), \tag{2}$$

where a is the spin dimension ($a=\frac{1}{2}(2-d-\eta)$ and η is the pair correlation function critical index). Now, recall that the two-dimensional Z_4 spin model is equivalent to the Ashkin–Teller model which consists of two coupled Ising models. This fact justifies the possibility of working in the neighbourhood of v, v', u and u' equal to zero. The whole procedure is extended without further justification to the three-dimensional case. Then, by linearising the fundamental equations (2) in v, v', u and u', and assuming that

$$v' = 2^{-a} v, \qquad u' = 2^{-2a} u, \tag{3}$$

after taking $h_1 = h_2 = 0$, the following recursion relations are obtained:

$$J_1' = (kJ_1/S)[e^{4J_2}(e^{4J_1}+1) + 2\,e^{-4J_2} + 7\,e^{2J_1} + e^{-2J_1} + 8],$$
$$J_2' = (kJ_2/S)[e^{4J_2}(e^{4J_1}+6+e^{-4J_1}) + 2(e^{J_1}+e^{-J_1})^2], \tag{4}$$

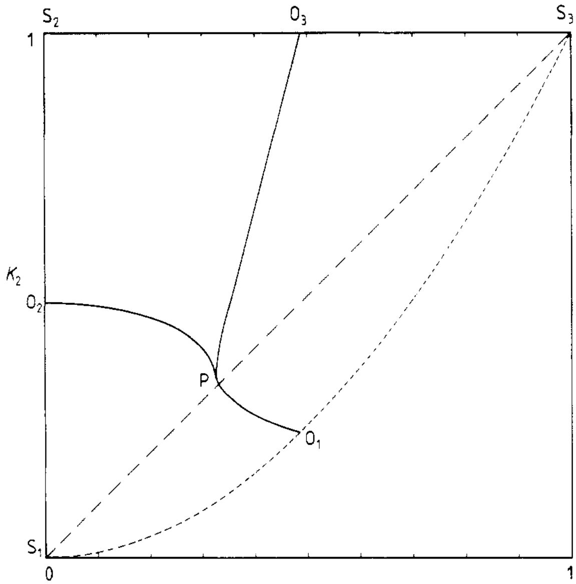
where

$$S = e^{4J_2}(e^{4J_1}+6+e^{-4J_1}) + 8\,e^{-4J_2} + 12(e^{J_1}+e^{-J_1})^2$$

and the constant k takes the value 2 in two dimensions and $\frac{8}{3}$ in three dimensions. Naturally, the direct identification of the values (v', u') with the functions (f', g') would lead to the usual MF results. Similarly, by equating (v, u) with (f, h) one would obtain an improved MF approximation which takes into account fluctuations inside the clusters.

The recursion relations can be rewritten in terms of the variables $K_1 = \exp(-2J_2 - J_1)$ and $K_2 = \exp(-2J_1)$. In the (K_1, K_2) space this transformation possesses the known (Ruján *et al* 1981, Creutz and Roberts 1983) seven fixed points of which (see figures 1 and 2) S_1, S_2 and S_3 are stable, O_1, O_2 and O_3, the Ising-like points, are stable in one direction and the triple point P, on the Potts line $K_1 = K_2$, is unstable. The flow of trajectories in the (K_1, K_2) space shows the presence of three phases characterised by (I): $v=u=0$, (II): $v\neq 0$, $u\neq 0$, (III): $v=0$, $u\neq 0$. In general, this MF–finite size scaling RG method reproduces the main qualitative features but it loses the self duality of the phase diagram.

The estimation of the critical points is much better than that of other methods of similar complexity, for example, the RSRG used by Knops (1975) in $d=2$ or the Migdal–Kadanoff RG with rescaling factor $\lambda = 2$. In particular, in the two-dimensional case, the triple point takes the exact value. In $d=3$, in spite of the anisotropy of the adopted cluster, the Ising-like and the triple points are determined within 5% with respect to the accepted values. This improvement in the determination of the Ising-like points with the dimension is a consequence of the underlying MF hypothesis that leads to the recursion relations (4).



12065 **Figure 1.** Phase diagram of the two-dimensional Z_4 spin model for $J_2 > 0$.

The critical exponents can be computed from the eigenvalues of the matrix $(\partial J_i/\partial J_j)(\boldsymbol{J}^*)$ obtained by linearising the transformation (4) around the fixed points $\boldsymbol{J}^*$. The largest eigenvalue, for each fixed point, can be written as $\lambda_1 = 2^{y_1}$ where y_1 is the thermal exponent related to the specific heat critical index $\alpha = 2 - d/y_1$.

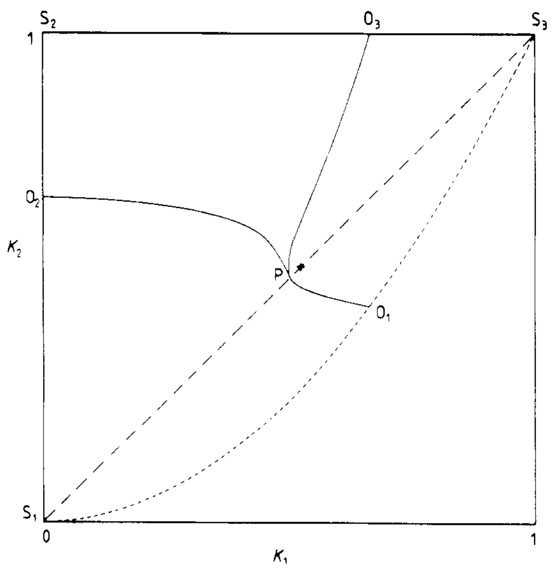


Figure 2. Phase diagram of the three-dimensional Z_4 spin model for $J_2 > 0$. The cross indicates the critical line of the four-state Potts model ($K_1 = K_2$ line).

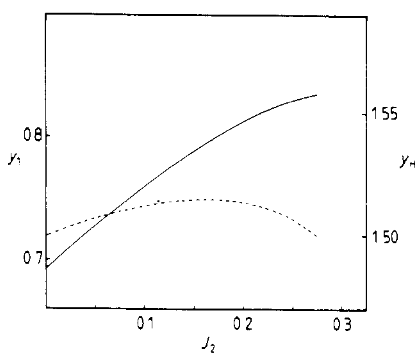


Figure 3. Critical exponents y_1 (full line) and y_H (broken line) against J_2 along the line PO_1.

In two dimensions, for the Ising-like fixed points, the value $y_1 = 0.693$ results, which yields a qualitatively wrong, negative α.

The same fact occurs for the triple point where the value $y_1 = 0.836$ is obtained. Its eigenvector is, as expected, in the direction of the Potts line.

The exponent y_H relative to the external field h_1 can be computed through the hyperscaling law $y_H = a - d$. For all the non-trivial fixed points, in $d = 2$, it results that $y_H = 1.5$.

By plotting the function

$$B = (G_1^2 + G_2^2)^{-1/2}$$

where $G_i = (K_i'(K_1, K_2) - K_i)$, it can be seen that the line PO_1 behaves as a line of fixed points. Then one can calculate the exponents y_1 and y_H along this line. The results are displayed in figure 3.

The calculation of the critical exponents in $d = 3$ has not been performed owing to the anisotropy of the chosen cluster. The computations for a cube or clusters of bigger size involves great difficulties that, for Z_4 variables, become almost prohibitive. Perhaps these difficulties can be overcome with the help of, for example, Monte Carlo techniques. Then it would be possible to test the convergence of the method for increasing cluster sizes.

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